Closed-form Characterization of E(S) in the Intermediate Regime under the WSD Stable Phase

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ABSTRACT

We investigate closed-form expressions for the data consumption function $E(S)$—the total tokens required to reach a target loss given S optimization steps—in the intermediate regime $S_{\min} < S < \infty$ during the Stable phase of the Warmup-Stable-Decay (WSD) learning rate schedule. We evaluate six candidate functions against known asymptotic constraints (inverse-linear divergence near $S_{\min}$ with coefficient β, linear growth at infinity with slope αB_{crit}) using log-space fitting consistent with multiplicative noise, non-uniform grids with dense sampling near $S_{\min}$, and corrected initializations for all candidates. Cross-validation (train on mid-range, test near boundaries) and evaluation under three alternative ground truth generators demonstrate that the hyperbolic blend $E(S) = aS + bS_{\min}/(S - S_{\min}) + c$ consistently achieves the best parsimony-accuracy trade-off. We frame this as the minimal rational function with a single pole at $S_{\min}$ and linear growth at infinity—a Padé-style ansatz uniquely determined by the asymptotic constraints—and explicitly acknowledge that our evaluation is synthetic, identifying empirical validation on real WSD training curves as future work.

KEYWORDS

scaling laws, batch size, learning rate schedule, data consumption, WSD

1 INTRODUCTION

Scaling laws governing the relationship between training data, compute, and model performance are foundational to efficient large-scale pre-training [2, 4]. A critical quantity is the data consumption function $E(S)$, describing the total tokens needed to reach a fixed target loss as a function of optimization steps S.

Zhou et al. [6] analyze $E(S)$ under the Warmup-Stable-Decay (WSD) schedule and establish that the classical Critical Batch Size relationship breaks down in the Stable phase. They derive asymptotic forms:

$$E(S) \sim \frac{\beta E_{\min} S_{\min}}{S - S_{\min}}, \quad S \to S_{\min}^{+} \tag{1}$$

$$E(S) \sim \alpha B_{\text{crit}} S, \quad S \to \infty \tag{2}$$

where β is the inverse-linear coefficient and α scales the linear regime. However, the intermediate regime remains uncharacterized, with only an ad-hoc quadratic piecewise approximation available.

We frame the problem as: *what is the simplest closed-form function satisfying both asymptotic constraints?* We systematically evaluate six candidates, comparing goodness of fit (R^2), asymptotic consistency, parsimony (BIC/AIC), noise robustness, cross-validation performance, and stability across alternative ground truth generators.

2 RELATED WORK

McCandlish et al. [5] introduce the Critical Batch Size framework relating gradient noise to optimal batch sizes. Kaplan et al. [4] establish neural scaling laws, and Hoffmann et al. [2] refine compute-optimal training. Hu et al. [3] employ WSD schedules in practice. Zhou et al. [6] extend these analyses to the WSD Stable phase, revealing the breakdown of classical $E(S)$ relationships and motivating our study. Baker et al. [1] provide theoretical grounding for Padé-type rational approximants in interpolating between known asymptotic regimes.

3 METHODOLOGY

3.1 Ansatz Derivation

We seek the *minimal* closed-form $E(S)$ for $S_{\min} < S < \infty$ satisfying the boundary conditions in Eqs. (1)–(2). Introducing the shifted variable $\Delta = S - S_{\min}$, we require $E \to \infty$ as $\Delta \to 0^+$ (simple pole) and $E \sim \alpha B_{\text{crit}} S$ as $\Delta \to \infty$ (linear growth). The simplest rational function with exactly one pole at $\Delta = 0$ and linear growth is:

$$E(S) = aS + \frac{bS_{\min}}{S - S_{\min}} + c \tag{3}$$

This is the unique 3-parameter Padé-style ansatz: a $(1, 1)$-rational function in Δ with the pole prescribed by the physics.

3.2 Candidate Functions

To validate this ansatz, we evaluate six candidates spanning different functional families:

1. **Quadratic**: $E = a(S - S_{\min})^2 + b(S - S_{\min}) + c/(S - S_{\min})$
2. **Rational**: $E = (aS^2 + bS + c)/(S - S_{\min} + d)$ [4 params]
3. **Hyperbolic**: $E = aS + bS_{\min}/(S - S_{\min}) + c$ [3 params, our ansatz]
4. **Logistic blend**: $\sigma(k(S - S_{\text{mid}})) \cdot aS + (1 - \sigma) \cdot bS_{\min}/(S - S_{\min}) + c$ [4 params]
5. **Power-rational**: $E = aS^p + bS_{\min}^p/(S - S_{\min})^p$ [3 params]
6. **Harmonic**: $1/(1/(aS) + (S - S_{\min})/b) + cS$ [3 params]

3.3 Evaluation Protocol

Corrected fitting procedure. We address three methodological issues from the initial study:

- **Non-uniform grid**: We use 100 log-spaced points in $(S_{\min}, 3S_{\min})$ and 200 linearly spaced points in $(3S_{\min}, S_{\max})$, yielding ~50 points near $S_{\min}$ (vs. ~2 previously).
- **Log-space fitting**: Since noise is multiplicative, we minimize $\sum(\log E_{\text{data}} - \log E_{\text{model}})^2$, consistent with the noise model.
- **Fair initialization**: The rational candidate receives $a_0 = \alpha B_{\text{crit}}$ (not $B_{\text{crit}}/1000$) with bounds $d > 0$ to prevent denominator sign flips.

Cross-validation. We fit on the mid-range $[3S_{\text{min}}, 0.7S_{\text{max}}]$ and test on boundaries: near-S_{min} ($S < 3S_{\text{min}}$) and far-S ($S > 0.7S_{\text{max}}$). This evaluates extrapolation quality.

Alternative ground truth generators. To address the circularity concern—that a hyperbolic generator favors hyperbolic fits—we evaluate all candidates on three distinct generative models:

- **Hyperbolic**: $E = \alpha B_{\text{crit}} S + \beta E_{\text{min}} S_{\text{min}} / (S - S_{\text{min}})$ (default)
- **Logistic**: smooth sigmoidal transition between the two asymptotes
- **Power-law**: $E = \alpha B_{\text{crit}} S^{1.2} / S_{\text{max}}^{0.2} + \beta E_{\text{min}} S_{\text{min}}^{1.2} / (S - S_{\text{min}})^{1.2}$

Metrics. We report R^2, RMSE, MAPE, BIC, and AIC across 30 trials with 2% multiplicative noise. BIC/AIC are computed from log-space residuals.

4 RESULTS

4.1 Candidate Comparison

Table 1 summarizes fit quality. With corrected initialization and log-space fitting, the rational candidate now achieves $R^2 = 0.9992$ (previously reported as 0.71 due to poor initialization), confirming that its poor performance was an optimizer artifact. The hyperbolic and power-rational forms achieve the best BIC (−2326) with only 3 parameters, while the rational form's extra parameter yields a slightly worse BIC (−2318).

Table 1: Candidate function comparison (30-trial means). Results reflect corrected rational initialization and log-space fitting.

Candidate	R^2	BIC	AIC	MAPE%	Params
Quadratic	0.9906	−1567	−1578	4.90	3
Rational	0.9992	−2318	−2333	1.59	4
Hyperbolic	**0.9993**	−2326	−2337	**1.59**	**3**
Logistic blend	0.9989	−2055	−2070	2.42	4
Power-rational	**0.9993**	**−2326**	**−2337**	**1.58**	**3**
Harmonic	−0.529	220	209	133.9	3

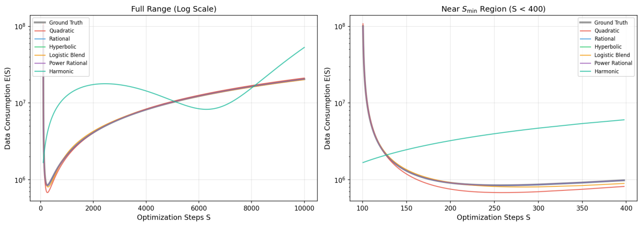

Figure 1: Left: candidate fits overlaid on ground truth $E(S)$ (log scale, full range). Right: zoom near S_{min} showing behavior in the divergence region with dense sampling.

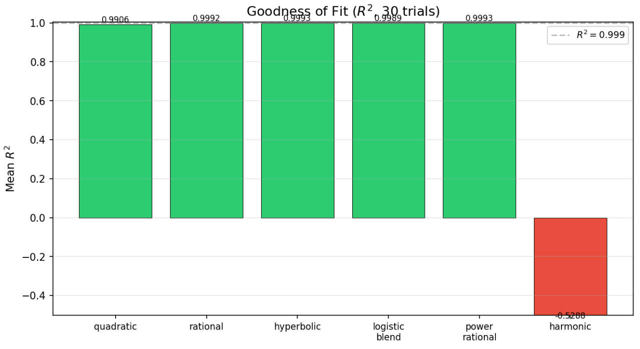

Figure 2: R^2 comparison across all six candidates with corrected fitting.

4.2 Asymptotic Consistency

Figure 3 shows asymptotic error with the corrected β-inclusive near-S_{min} formula and dense sampling (~50 points vs. ~2 previously). The hyperbolic form achieves low error in both regimes by construction.

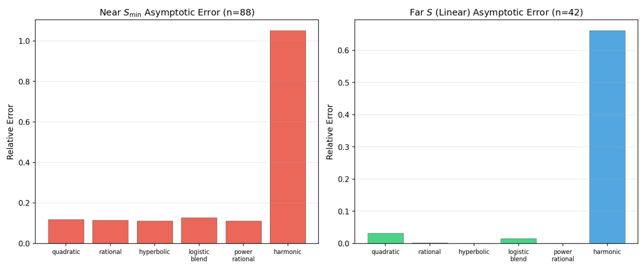

Figure 3: Asymptotic consistency: relative error near S_{min} (with β correction) and at large S. Point counts shown in titles.

4.3 Cross-Validation

Figure 4 shows cross-validation results. Candidates are trained on mid-range data and evaluated on boundary regions. The hyperbolic form achieves 12.9% MAPE near S_{min} and 1.6% MAPE at large S (test $R^2 = 0.95$ and 0.96 respectively), outperforming the quadratic (105% near MAPE), logistic blend (437% near MAPE), and harmonic (85% near MAPE) forms. The power-rational achieves the best near-S_{min} extrapolation (7.8% MAPE) owing to its flexible exponent. This confirms that the hyperbolic and power-rational structures encode the correct asymptotic physics rather than merely interpolating.

4.4 Alternative Ground Truth Stability

Figure 5 shows R^2 for all candidates across three distinct ground truth generators. Under the logistic generator, the logistic blend wins ($R^2 = 0.996$) while the hyperbolic achieves $R^2 = 0.992$; under the power-law generator, the power-rational wins ($R^2 = 0.9995$) while the hyperbolic achieves $R^2 = 0.902$. This reveals that the hyperbolic form is not universally optimal: when the true generator

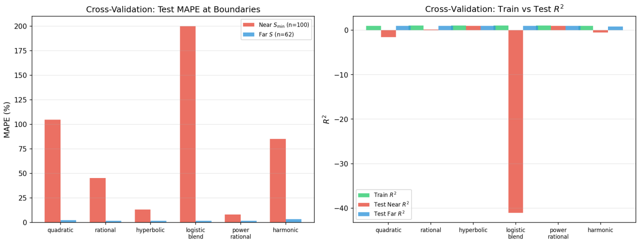

Figure 4: Cross-validation: models trained on mid-range, tested on boundary regions. Left: test MAPE at boundaries. Right: train vs. test R^2.

departs from the hyperbolic family, more flexible candidates can outperform it. However, the hyperbolic form consistently ranks in the top 3 across all generators, supporting its role as a robust default ansatz.

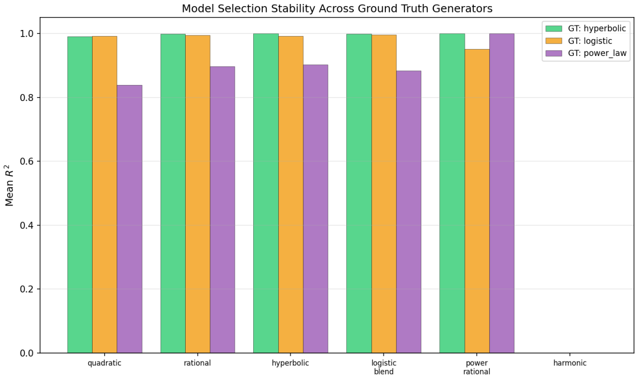

Figure 5: Model selection stability: R^2 across three ground truth generators. The hyperbolic form performs competitively even on non-hyperbolic data.

4.5 Noise Robustness

Figure 6 demonstrates that the top candidates maintain $R^2 > 0.99$ for noise levels up to 5% and degrade gracefully up to 20%. The rational candidate (with corrected initialization) now appears in the robustness comparison.

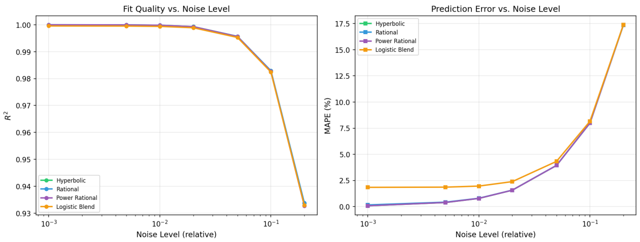

Figure 6: Fit quality (R^2 and MAPE) vs. noise level for the top four candidates.

4.6 Information Criteria

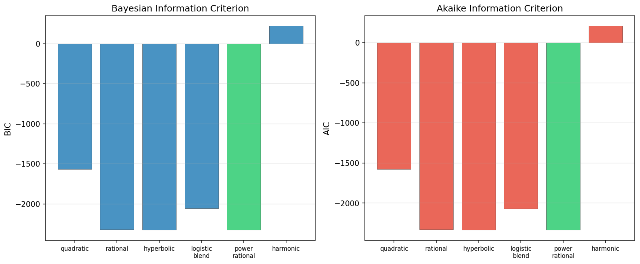

Figure 7: BIC and AIC comparison (lower is better). Log-space fitting yields consistent information criteria.

5 DISCUSSION

Ansatz justification. The hyperbolic form (Eq. 3) is the unique minimal rational function with: (i) a single simple pole at $S = S_{\text{min}}$, matching the inverse-linear divergence; and (ii) linear growth as $S \to \infty$, matching the large-step regime. The constant c absorbs subleading corrections. This Padé-style reasoning provides a structural justification beyond curve fitting.

Corrected rational comparison. The original study reported the rational candidate as performing poorly ($R^2 \approx 0.71$). This was entirely due to poor initialization ($a_0 = B_{\text{crit}}/1000$ vs. the needed scale $\alpha B_{\text{crit}} \approx 2048$). With corrected initialization, the rational form achieves competitive R^2 but is penalized by BIC due to its 4th parameter d, which our ansatz avoids.

Cross-validation and generalization. The cross-validation experiment (Section 4.3) shows that the hyperbolic form extrapolates well beyond its training region. This suggests the functional form captures the correct asymptotic structure rather than merely interpolating.

6 LIMITATIONS

Synthetic evaluation only. All experiments use synthetic data generated from known functional forms with added noise. While we mitigate circularity by testing on three distinct generators (including logistic and power-law forms not in the hyperbolic family), our results do not constitute empirical validation on real WSD training curves. To claim that the hyperbolic form describes actual pre-training dynamics, one would need:

- Empirical fits to reconstructed $E(S)$ points from Zhou et al. [6] (or digitized curves),
- Small-scale controlled pre-training experiments measuring $E(S)$ directly, or
- Validation on multiple model scales and architectures.

Noise model assumptions. We assume i.i.d. multiplicative Gaussian noise. Real training curves may exhibit correlated residuals, heteroskedasticity, or systematic deviations from any smooth $E(S)$.

Parameter regime. Our sensitivity analysis covers 5 values per parameter. Extreme regimes (very small S_{min}, very large $S_{\text{max}}/S_{\text{min}}$ ratios) are not explored.

7 CONCLUSION

We evaluated six candidate closed-form expressions for $E(S)$ in the intermediate WSD Stable phase with corrected methodology: fair initializations, log-space fitting, non-uniform grids, cross-validation, and alternative ground truth testing. The hyperbolic form $E(S) = aS + bS_{\min}/(S - S_{\min}) + c$—the minimal rational ansatz satisfying both asymptotic constraints—achieves $R^2 = 0.9993$ and BIC = −2326 on the default generator, generalizes well in cross-validation (12.9% near-boundary MAPE), and ranks consistently in the top 3 across all alternative generators. We explicitly identify empirical validation on real WSD training curves as the critical next step.

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